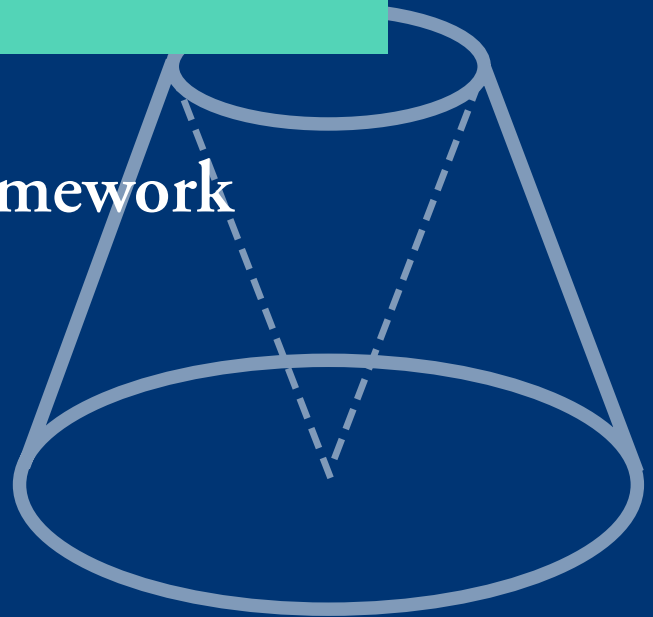




Mathematics Framework for California Public Schools

Kindergarten Through
Grade Twelve



$$E=mc^2$$



$$1 + 2 = 3$$

$$8 + (\quad) = 13$$

Mathematics Framework for California Public Schools

Kindergarten Through
Grade Twelve

Developed by the
Curriculum Development and Supplemental Materials Commission

Adopted by the
California State Board of Education

Published by the
California Department of Education



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A partial list of other educational resources available from the Department appears on page 395. In addition, an illustrated *Educational Resources Catalog* describing publications, videos, and other instructional media available from the Department can be obtained without charge by writing to the address given above or by calling the Sales Office at (916) 445-1260.

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Foreword

Pure mathematics is, in its way, the poetry of logical ideas.

Albert Einstein (1879–1955)

In 1997, with the adoption of the *Mathematics Content Standards for California Public Schools*, California provided a foundation for a high-quality mathematics program that prepares all students for postsecondary education and a wide range of careers. This updated edition of the *Mathematics Framework for California Public Schools, Kindergarten through Grade Twelve* is based on the standards and provides a context for continuing a coordinated effort to enable all California students to achieve rigorous, high levels of mathematics proficiency.

This framework focuses squarely on the content standards and calls for instructional programs and strategies, instructional materials, professional development, and assessments that are aligned with the standards. This edition also includes the evaluation criteria that will guide the primary adoption of mathematics instructional materials (for kindergarten through grade eight) to be conducted in 2007. The evaluation criteria for grade-level mathematics programs requires alignment with the appropriate grade-level content standards for kindergarten through grade seven and the Algebra I standards for grade eight.

Over the last several years, mathematics student achievement in California has improved. More students are performing at proficient or above levels in most grades and many courses, and there has been a significant increase in the number of students taking the more college preparatory mathematics courses such as algebra and geometry.

The framework highlights the importance of all students receiving a solid education in mathematics and successfully completing the state's high school graduation requirements for mathematics. Students must complete at least two courses in mathematics in grades nine through twelve (one or a combination of these courses must meet or exceed the rigor of the content standards for Algebra I).

Beginning with the class of 2006, state law also requires that all public school students pass the *California High School Exit Examination (CAHSEE)* in order to receive a high school diploma. For those students who are in danger of not mastering the mathematics skills measured by the *CAHSEE* and are struggling in their preparation for algebra, the framework outlines the design of instructional materials for algebra readiness (for students in grade eight or higher who are not prepared for algebra) and mathematics intervention (for students in grades four through seven). These materials may be included on the list of mathematics instructional materials for kindergarten through grade eight adopted by the

State Board of Education beginning in the fall of 2007 and will provide districts with additional choices in selecting instructional materials to help all students successfully complete Algebra I.

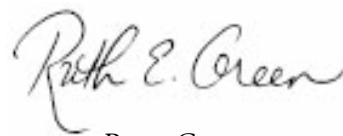
In addition, the framework provides instructional guidance for teachers to enable them to help students achieve mastery of the mathematics standards. Sample problems illustrate the standards and discuss common misconceptions and pitfalls regarding standards-based education, providing ways to ensure that students are ready to move from grade to grade and discipline to discipline. The framework stresses careful and thoughtful sequencing of instruction based on the standards so that students will master basic skills and techniques for sophisticated problem solving and understand sound mathematical reasoning.

Mathematics—using abstract symbols to describe, order, explain, and predict—has become essential to human existence. It is a universal language of numbers that is spoken throughout the world and is as important as oral and written language is in the communication of ideas. Young people who have taken gateway courses such as algebra and geometry go on to college at much higher rates than those who do not. The study of mathematics has contributed to the fields of medicine, management, economics, government, computer science, physics, psychology, engineering, and social science. Mathematics also enables us to conduct the simple yet vital transactions of daily life, such as telling time, gauging distances, and making change.

To ensure that tomorrow's leaders are equipped to compete in our global economy and are able to make connections between mathematics and everyday life, California needs to educate all students in mathematics at levels consistent with their counterparts throughout the world. This framework maps the route toward that very achievable goal as set forth in the *Mathematics Content Standards for California Public Schools*. It is a journey that we have no choice but to travel and no alternative but to complete successfully.



JACK O'CONNELL
State Superintendent of Public Instruction



RUTH GREEN
President, State Board of Education

Acknowledgments

The updated 2005 edition of the *Mathematics Framework for California Public Schools*, which includes revised “Criteria for Evaluating Mathematics Instructional Materials, Kindergarten Through Grade Eight” (Chapter 10), was adopted by the California State Board of Education on March 9, 2005. When the 2005 edition of the framework and the criteria were approved, the following persons were serving on the State Board:

Ruth E. Green, President
Glee Johnson, Vice President
Ruth Bloom, Member
Donald G. Fisher, Member
Joe Nuñez, Member
Bonnie Reiss, Member
Johnathan Williams, Member
Ricky Gill, Student Member

Members of the Curriculum Development and Supplemental Materials Commission (Curriculum Commission) serving in January 2005, when the 2005 edition of the framework and the criteria were recommended for approval by the State Board, were:

Norma Baker, Chair, Los Angeles Unified School District
Deborah Keys, Vice Chair, Oakland Unified School District
William Brakemeyer (Retired), Fontana Unified School District
Assembly Member Jackie Goldberg, California State Assembly
Kerry Hamill, Oakland Unified School District
Wendy Levine, Inglewood Unified School District
Sandra Mann, San Diego City Unified School District
Julie Maravilla, Los Angeles Unified School District
Michael Matsuda, Anaheim Union High School District
Mary-Alicia McRae, Salinas City Elementary School District
Stan Metzenberg, California State University, Northridge
Charles Munger, Jr., Palo Alto, California
Armida Ornelas, East Los Angeles College
Rebecca (Brown) Sullivan, Sacramento County Office of Education
Senator John Vasconcellos, California State Senate
Joseph Velasquez, Los Angeles Unified School District
Richard Wagoner, Los Angeles Unified School District

Commissioners who also served on the Mathematics Subject Matter Committee (SMC) of the Curriculum Commission, with oversight for the 2005 revision of the framework and the criteria during 2004–05, were:

Norma Baker, Chair, Los Angeles Unified School District
Charles Munger, Jr., Vice Chair, Palo Alto, California
Assembly Member Jackie Goldberg, California State Assembly
Wendy Levine, Inglewood Unified School District
Stan Metzenberg, California State University, Northridge
Richard Wagoner, Los Angeles Unified School District

In addition, a special thanks is extended to former commissioner and past Chair of the Curriculum Commission, **Edith Crawford**, San Juan Unified School District, for her contributions to the development of this framework, including soliciting the help of experts in mathematics education from her district.

Appreciation is also expressed to **Sue Stickel**, Deputy Superintendent, Curriculum and Instruction Branch, California Department of Education, for her leadership and expertise in mathematics education that helped guide the development of this framework.

For the 2005 edition of the framework, the following managers coordinated the procedures for publishing the document and the overall process for incorporating legislative changes, mathematical corrections, recommendations from the education community and the general public, and other updated material:

Thomas Adams, Director, Curriculum Frameworks and Instructional Resources Division, and Executive Director, Curriculum Commission
Donald Kairott, Administrator, Curriculum Frameworks Unit

The following staff members from the Curriculum Frameworks and Instructional Resources Division contributed to the process of completing the 2005 edition of the framework:

Tom Akin, Visiting Educator
Christine Bridges, Analyst
Jim Long, Analyst
Teri Ollis, Analyst
Patrice Roseboom, Assistant Consultant
Mary Sprague, Consultant

Other California Department of Education staff who contributed mathematics expertise during the development of the 2005 edition were:

Yvonne Evans, Consultant, Mathematics/Science Leadership Unit
Kay Garcia, Consultant, Mathematics/Science Leadership Unit
Tom Lester, Retired Annuitant, Curriculum Frameworks Unit
Bill Tarr, Visiting Educator, Standards and Assessment Division

Gratitude is expressed to **James Milgram**, Stanford University, and **Vik Hovsepian**, Rio Hondo College, for developing initial drafts of Appendix E, which provides information on the design of mathematics intervention and algebra readiness instructional materials.

Appreciation is extended to the following persons who reviewed sections of the draft framework for mathematical accuracy:

Paul Clopton, Veterans Affairs Medical Center, San Diego

David Klein, California State University, Northridge

Several educators made presentations before the Curriculum Commission and offered public comments on the need for and design of mathematics intervention programs and materials, including:

Nancy Balhorn, Mira Loma High School

Pat Bergen, Mira Loma High School

Pat Duckhorn, Sacramento County Office of Education

Susan Greene, San Juan Unified School District

Chris King, Stanislaus County Office of Education

Jodie Thayer, Mira Loma High School

Contributions to the development of Appendix E from the following representatives from the California Mathematics Council were greatly appreciated:

Harold Asturias, Mike Contino, Katy Early, Diana Herrington, Sara Munshin, Lisa Usher, and Vickie Vierra.

The Curriculum Frameworks Unit thanks **Char Keuscher-Barkman** from the California Association for Supervision and Curriculum Development (ASCD) for organizing and hosting a series of videoconferences to solicit public comments on the framework during the field review.

The information that follows relates to the previous edition of the *Mathematics Framework*, adopted in December 1998. (The titles and employment status of the persons listed in this section were current in 1998 and may not reflect present titles or employment.)

The following State Board members supported the Curriculum Commission and provided leadership in the development of the framework that was adopted in 1998:

Yvonne W. Larsen, President, State Board of Education

Robert L. Trigg, Vice President

Marian Bergeson, Member

Timothy Draper, Member

Kathryn Dronenburg, Member

Marion Joseph, Member

Marion McDowell, Member

Janet G. Nicholas, Member

Gerti B. Thomas, Member

Marina Tse, Member

Richard Weston, Member

A special thanks is also extended to **Robert L. Trigg** and **Janet G. Nicholas**, liaisons for mathematics from the State Board during the development of the 1998-adopted framework.

Curriculum Commission members during the development of the 1998-adopted framework were:

Kirk Ankeney, Chair, Curriculum Commission, Vice Principal, San Diego Unified School District

Patrice Abarca, Teacher, Los Angeles Unified School District

Roy Anthony, Teacher, Grossmont Union High School District

Marilyn Astore, Assistant Superintendent, Instructional Support Services, Sacramento County Office of Education

Eleanor Brown, Assistant Superintendent, San Juan Unified School District

Ken Dotson, Teacher, Turlock Joint Elementary School District

Michele Garside, Superintendent, Laguna Salada Union School District

Senator Leroy F. Greene, California State Senate

Viken “Vik” Hovsepian, Teacher/Chair, Mathematics Department, Glendale High School, Glendale Unified School District

Lisa Jeffery, Vice Principal, Los Angeles Unified School District

Joseph Nation, Managing Director, California Data & Analysis, San Rafael

Richard Schwartz, Teacher, Torrance Unified School District

Assembly Member Jack Scott, California State Assembly

Barbara Smith, Superintendent, San Rafael City Schools

Sue Stickel, Assistant Superintendent, Instructional Support, Elk Grove Unified School District

Lillian Vega Castaneda, Associate Professor, California State University, San Marcos

Jean Williams, Retired Area Assistant Administrator, Fresno Unified School District

Commissioners who also served as members of the Mathematics SMC during the development of the 1998-adopted framework were:

Barbara Smith, Math SMC Chair, Superintendent, San Rafael City Schools

Marilyn Astore, Assistant Superintendent, Instructional Support Services, Sacramento County Office of Education

Michele Garside, Superintendent, Laguna Salada Union School District

Viken “Vik” Hovsepian, Teacher/Chair, Mathematics Department, Glendale High School, Glendale Unified School District

Richard Schwartz, Teacher, Torrance Unified School District

Sue Stickel, Assistant Superintendent, Instructional Support, Elk Grove Unified School District

The 1998-adopted framework reflects the work of a number of university professors. The following university mathematics professors developed sample problems to illustrate the mathematics standards included in that framework:

Ralph Cohen, Stanford University

Scott Farrand, California State University, Sacramento

H-H Wu, University of California, Berkeley

David Geary, Cognitive Psychologist from the University of Missouri, prepared information on how students learn mathematics, the models for mathematics instruction, and the key standards at each grade level for the 1998-adopted framework.

Shin-ying Lee, Center for Human Growth and Development, University of Michigan, provided Appendix B, “Elementary School Sample Lesson: An East Asian Approach,” for the 1998 and 2005 editions of the framework.

In addition, **H-H Wu**, University of California, Berkeley, and **James Milgram**, Stanford University, performed a technical review of the 1998-adopted framework to ensure mathematical accuracy throughout the document.

Initial drafts of the 1998-adopted *Mathematics Framework* were developed by the Mathematics Curriculum Framework and Criteria Committee (CFCC). A partial list of the committee membership follows:

Deborah Tepper Haimo, CFCC Chair, University of California, San Diego

Henry Alder, University of California, Davis

Ruth Uy Asmundson, School Board Trustee, Davis Joint Unified School District

Eduardo Chamorro, Los Banos Unified School District

Paul Clopton, Veterans Affairs Medical Center, San Diego

Ralph Cohen, Stanford University

Mike Erickson, Santa Cruz City Schools

Viken “Vik” Hovsepian, Glendale Unified School District

Grace Hutchings, Los Angeles Unified School District

Alla Korinevskaya, Temple Emanuel Day School

Calvin Moore, University of California, Berkeley

Martha Schwartz, California State University, Dominguez Hills

Dennis Stanton, Santa Cruz City Schools

Bob Stein, California State University, San Bernardino

Paraskevi Steinberg, Sacramento City Unified School District

Ze’ev Wurman, Dyna Logic, Santa Clara

Subsequent to the development of the initial drafts of the 1998 *Mathematics Framework*, the State Board of Education adopted content standards in mathematics (December 1997). The Curriculum Commission then revised the draft version of the framework to incorporate the mathematics standards.

The principal writer for the 1998-adopted framework, working both with the CFCC and with the Curriculum Commission members at that time, was **Hank Resnik**.

California Department of Education staff members who contributed to the process of developing the 1998-adopted framework were:

Leslie Fausset, Chief Deputy Superintendent for Educational Policy, Curriculum, and Department Management

Sonia Hernandez, Executive Secretary to the Curriculum Commission and Deputy Superintendent, Curriculum and Instructional Leadership Branch

Wendy Harris, Assistant Superintendent and Director, Elementary Education Division

Catherine Barkett, Administrator, Curriculum Frameworks and Instructional Resources (CFIR) Office

Tom Adams, Consultant, CFIR Office

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Joanne Knieriem, Support Staff, CFIR Office

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Mary Sprague, Consultant, CFIR Office

Tracy Yee, Analyst, CFIR Office

Appreciation is also extended to the following persons for their assistance with the development of the 1998-adopted framework:

Susan Greene, Math Specialist, San Juan Unified School District

Nancy Maiello, Teacher, Mira Loma High School, San Juan Unified School District

Introduction

All students need a high-quality mathematics program designed to prepare them to choose from among a full range of career paths. California's Mathematics Task Force (1995) called for a rigorous and challenging mathematics program for every student—a complete program that reflects a balance of computational and procedural basic skills, conceptual understanding, and problem solving. Although California students have made gains in recent years, the mathematics achievement of many students remains unacceptably low (NCES 2003). Educators are challenged to expect more from students in mathematics, to raise the bar for student achievement through more rigorous curriculum and instruction, and to provide the support necessary for all students to meet or exceed California's mathematics standards.

To compete successfully in the worldwide economy and to participate fully as informed citizens, today's students must have a high degree of comprehension of mathematics. The goal of the California State Board of Education is that all students will attain California's mathematics standards for kindergarten through grade twelve (hereinafter referred to as “the standards”) and that many will be inspired to pursue advanced studies in mathematics.

This framework is predicated on the belief that proficiency in mathematics is a consequence of sustained student effort and effective instruction. All students are capable of understanding mathematics, given the opportunities and encouragement to do so.

What's New in the 2005 Framework?

The most important feature of this framework is its continued focus on the content of the mathematics standards adopted by the California State Board of Education in 1997. The goal of accelerating students' progress through a standards-based program has a significant impact on the entire curriculum. As specified by the standards, much of the content of the mathematics curriculum has shifted into earlier grades, and mathematics instruction in kindergarten through grade seven has been substantially strengthened. These standards necessitate a more continued substantive, rigorous, and demanding curriculum and systematic instruction to better serve California students.

A goal of this framework is to prepare all students to study algebra by the eighth grade. In 2000 legislation was enacted requiring students to complete Algebra I as a condition of receiving a high school diploma (*Education Code* Section 51224.5(b)). To support this legislative goal for all students, the framework includes guidance on the design of two new types of specialized instructional mathematics materials for students having difficulty achieving at grade level: a mathematics intervention program for students in grades four

through seven whose mastery of prior grade-level standards is incomplete and algebra readiness materials for students in grade eight or above who are not prepared to take a course in algebra. If, when, and how these materials may be used, to meet the needs of students, should be a collective decision involving teachers, counselors, administrators, parents, and students. Intervention materials are intended to accelerate students' achievement so that students can succeed in the basic grade-level program and complete Algebra I. As previously stated, the study of Algebra I in the eighth grade is the goal. The ability of districts to purchase and use algebra readiness materials should not restrict students who are ready for Algebra I in the eighth grade from enrolling in an Algebra I course.

An important theme stressed throughout this framework is the need for a balance in emphasis on computational and procedural skills, conceptual understanding, and problem solving. This balance is defined by the standards and is illustrated by problems that focus on these components individually and in combination. All three components are essential.

A General Overview: Purpose, Organization, and Audience

The purpose of this framework is to guide the curriculum development and instruction that teachers provide in their efforts to ensure that all students meet or exceed the mathematics standards. The framework provides a context for implementing the standards in the form of guidelines for the design of curricula, instructional materials, instructional practices, and staff development. Building on the standards, the framework addresses the manner in which all students in California public schools can best meet the standards. *All students* includes those performing at, below, and above grade level; English learners; special education students; and others with special learning needs.

More specifically, the framework:

- Describes *guiding principles and key components* of an effective mathematics program (Chapter 1)
- Presents the essential skills and knowledge expected of students in mathematics as described in the *Mathematics Content Standards for California Public Schools* and illustrated by sample problems (Chapter 2)
- Describes *special considerations and emphases for each grade level to ensure student success* consistent with the mathematics standards and statewide testing program (Chapter 3)
- Provides guidance, based on current research, regarding *instructional strategies* and sample lessons that can be effective in ensuring that every child will meet or exceed grade-level standards in mathematics (Chapter 4)
- Guides the development of appropriate *assessment* methods (Chapter 5)
- Suggests specific strategies to ensure *access* to appropriately challenging curriculum for special needs students (Chapter 6)

- Describes the *responsibilities* that all stakeholders must uphold for effective implementation of a rigorous and coherent kindergarten through grade twelve mathematics curriculum (Chapter 7)
- Suggests guidelines for both preservice teacher preparation and in-service *professional development* (Chapter 8)
- Provides guidance on the use of *technology* in mathematics instruction (Chapter 9)
- Specifies requirements for *instructional resources*, including print and electronic learning resources (Chapter 10)

In short, the framework should be viewed as a critical tool for designing and implementing an effective mathematics program in kindergarten through grade twelve and for evaluating instructional resources.

The framework addresses two primary audiences: (1) educators; and (2) developers of instructional resources. Educators include those involved in the day-to-day implementation of school mathematics programs—classroom teachers, school administrators, district personnel, school board members, and others responsible for curriculum and instruction. It also addresses other important audiences, such as parents and community members, including business and civic leaders, who have a vital stake in the success of California students in mathematics.

The following themes permeate the *Mathematics Framework*:

The framework:

- **Builds on the mathematics standards** and aligns them with curriculum, instruction, resources for instruction, and assessment, resulting in a coherent and pragmatic plan for achieving high levels of mathematics proficiency for all students. It also provides guidance for understanding the standards by identifying priorities within the standards and offering concrete examples of mathematical problems that demonstrate the concepts within the standards.

The framework:

- **Emphasizes the importance of a balanced mathematics curriculum.** In particular, the framework stresses the critical interrelationships among computational and procedural proficiency, problem-solving ability, and conceptual understanding of all aspects of mathematics, from the simplest calculations to the most sophisticated problem solving.

The framework:

- **Addresses the needs of all learners**, with no learner left out and no learner taught at the expense of another; emphasizes prevention over remediation, while recognizing the appropriateness of remediation when it is required; and provides suggestions for instructional strategies that may be used with students who are English learners, advanced learners, special education pupils, or at risk of failing mathematics.

The framework:

- **Highlights the importance of mathematical reasoning.** The mathematical reasoning standards are different from the others in that they do not represent a specific content area. Mathematical reasoning cuts across all strands. It characterizes the thinking skills that students can carry from mathematics into other disciplines. Constructing valid arguments and criticizing invalid ones are inherent in doing mathematics.

The framework:

- **Stresses the importance of frequently assessing student progress toward achieving the standards.** Students cannot afford to wait for a year-end test; rather, they should be assessed frequently throughout the year to determine their progress toward achieving the standards. Teachers, students, and parents and guardians need some daily or weekly indication of the extent to which the standards are likely to be met.

The framework:

- **Avoids oversimplified guidance on either content or pedagogy** in favor of guidelines on effective instruction derived from reliable research.